

# Comment on “Scaling behavior of classical wave transport in mesoscopic media at the localization transition”

N. Cherroret, S.E. Skipetrov, and B.A. van Tiggelen

*Université Joseph Fourier, Laboratoire de Physique et Modélisation des Milieux Condensés,  
CNRS UMR 5493, B.P. 166, 25 rue des Martyrs,  
Maison des Magistères, 38042 Grenoble Cedex 09, France*

(Dated: July 24, 2009)

We emphasize the importance of the position dependence of the diffusion coefficient  $D(\mathbf{r})$  in the self-consistent theory of localization and argue that the scaling law  $T \propto \ln L/L^2$  obtained by Cheung and Zhang [Phys. Rev. B **72**, 235102 (2005)] for the average transmission coefficient  $T$  of a disordered slab of thickness  $L$  at the localization transition is an artifact of replacing  $D(\mathbf{r})$  by its harmonic mean. The correct scaling  $T \propto 1/L^2$  is obtained by properly treating the position dependence of  $D(\mathbf{r})$ .

PACS numbers: 42.25.Dd

In a recent paper [1] Cheung and Zhang (CZ) apply the self-consistent (SC) theory of localization to study the transmission of waves through a slab of disordered medium at the Anderson localization transition. The SC theory is a powerful tool to deal with the phenomenon of Anderson localization, but its application to disordered media of finite size requires some care. In the original papers by Vollhardt and Wölfle [2], the size  $L$  of disordered sample was acknowledged using a lower cut-off in the integration over momentum. Despite the obvious crudeness of this approach, it was sufficient to recover the main results of the scaling theory of localization [3] and added a great physical insight into the phenomenon of disorder-induced localization. Later on, Van Tiggelen *et al.* [4] argued that in a medium of finite size the SC theory naturally leads to a position dependence of the diffusion coefficient  $D(\mathbf{r})$ . This adapted SC theory was successfully applied to study coherent backscattering [4] and dynamics [5, 6] of localized waves. Microscopic justifications for position dependence of  $D$  have been recently presented based on the diagrammatic [7] and field-theoretic [8] calculations.

CZ propose a way of overcoming technical difficulties caused by the position dependence of  $D(\mathbf{r})$  [1] (see also [9]). They average the equation for  $1/D(\mathbf{r})$ , their Eq. (1), over the sample volume, thus replacing  $D(\mathbf{r})$  by its harmonic mean  $\bar{D}$ . In this Comment we argue that although such an approach can be justified in the weak localization regime [9], it is not adequate at the mobility edge and in the Anderson localization regime. In particular, our calculations that properly treat the position dependence of  $D(\mathbf{r})$ , do not confirm the scaling law  $T \propto \ln L/L^2$  found by CZ for the transmission coefficient  $T$  of a disordered slab of thickness  $L$  at the mobility edge. Instead, we find  $T \propto 1/L^2$  in agreement with the scaling theory of localization [3].

To study the scaling of the average transmission coefficient  $T$  with the thickness  $L$  of disordered slab, we solve the two equations of SC theory — Eqs. (1) and (2) of Ref. [6] with  $\Omega = 0$  (stationary regime) and  $k\ell = 1$  (mobility edge) [10] — numerically. We use the same boundary

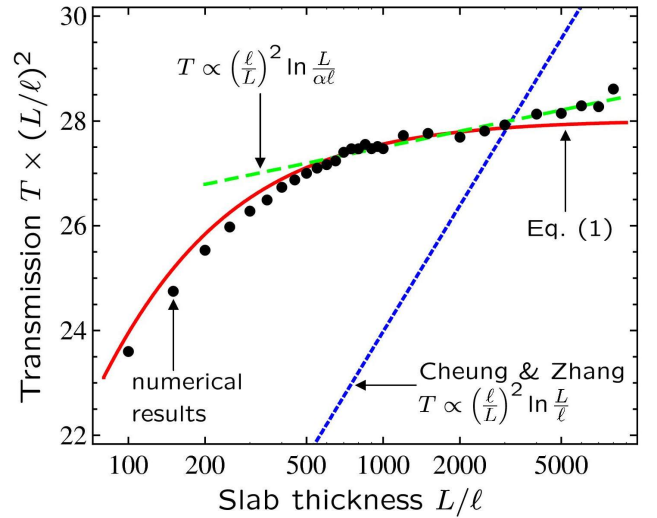


FIG. 1: Average transmission coefficient  $T$  of a disordered slab of thickness  $L$  at the Anderson localization transition. Circles were obtained from the self-consistent theory of localization with a position-dependent diffusion coefficient  $D(z)$  by numerical solution [6]. The solid red line is a fit to the numerical results using Eq. (1) with  $D(0)/D_B = 0.82$  and  $z_c = 4.2\ell$ . The dotted blue and dashed green lines are fits to numerical data for  $L/\ell > 10^3$  using  $T \propto (\ell/L)^2 \ln(L/\ell)$  and  $T \propto (\ell/L)^2 \ln(L/\alpha\ell)$ , respectively. We obtain  $\alpha \simeq 7.37 \times 10^{-25}$  in the latter case.

conditions and the same method of numerical solution as in Ref. [6] and vary the thickness of the slab  $L$  from  $10^2\ell$  to  $8 \times 10^3\ell$ . Here  $k$  is the wave number of the wave and  $\ell$  is the mean free path due to disorder. Our results are presented in Fig. 1 by circles. The red solid line in Fig. 1 shows

$$T = \left(\frac{\ell}{L}\right)^2 \frac{2 + 4\frac{z_c}{\ell} \left[1 + \frac{D(0)}{D_B} \frac{z_0}{\ell}\right]}{1 + 4\frac{z_c}{L} \left[1 + 2\frac{D(0)}{D_B} \frac{z_0}{L}\right]} \quad (1)$$

that we obtained by assuming  $D(z) = D(0)/(1 + \tilde{z}/z_c)$

with  $\tilde{z} = \min(z, L - z)$  as suggested by Van Tiggelen *et al.* [4]. Here  $D_B$  is the diffusion coefficient in the absence of macroscopic interferences (i.e. in the limit of  $k\ell \gg 1$ ).  $D(0)/D_B$  was determined directly from the numerical results at a sufficiently large  $L = 10^3\ell$ , whereas  $z_c$  was a free fit parameter. We used  $z_0 = \frac{2}{3}\ell$ , corresponding to no internal reflections at the sample boundaries. Deviations of the fit from the numerical results do not exceed 3% in the whole range of considered  $L$ 's, which supports the validity of Eq. (1) and its underlying model for  $D(z)$ . The inaccuracy of the latter model in the middle of the slab cause deviations at small  $L < 10^3\ell$ , whereas deviations at large  $L > 4 \times 10^3\ell$  are mostly due to the extremely slow convergence of our computational algorithm for thick slabs and would, most likely, disappear if more computer time were available. We note that  $T \times (L/\ell)^2$  grows with  $\ln(L/\ell)$  for  $L < 10^3\ell$ , but then saturates at a constant level for larger  $L$ , suggesting  $T \propto (\ell/L)^2$  in the limit of large  $L$ .

Neither the ensemble of numerical results of Fig. 1, nor its small- or large- $L$  parts can be fit by  $T = \text{const} \times (\ell/L)^2 \ln(L/\ell)$  proposed by CZ. This is easy to see from Fig. 1 where we show a fit of the above equation to our numerical data for  $L/\ell > 10^3$  (dotted blue straight line). It is clear that the fast growth of  $T \times (L/\ell)^2$  with  $\ln(L/\ell)$  predicted by CZ is not supported by our numerical calculations: the numerical results only show an increase of

20% in the range of  $L/\ell = 100$ –8000 and 4% in the range  $L/\ell = 1000$ –8000, whereas the result of CZ increases by 100% and 30%, respectively. For large  $L > 10^3\ell$ , a reasonable fit can be achieved by  $T \propto (\ell/L)^2 \ln(L/\ell)$ . The result of CZ would correspond to  $\alpha \sim 1$ , whereas a fit to the numerical data yields  $\alpha \sim 10^{-24} \ll 1$ . This value is unphysically small and implies existence of length scales that are 24 orders of magnitude shorter than the mean free path  $\ell$ . We therefore conclude that our numerical results exclude the possibility of logarithmic scaling of  $T \times (L/\ell)^2$  with  $L/\ell$ . Appearance of this scaling in Ref. [1] should then be an artifact of replacing  $D(\mathbf{r})$  by its harmonic mean.

In conclusion, we have shown the importance of properly treating the position dependence of the diffusion coefficient  $D(\mathbf{r})$  in the SC theory of localization. In particular, replacing  $D(\mathbf{r})$  by its harmonic mean leads to an incorrect scaling law for the transmission coefficient  $T$  with the thickness  $L$  of disordered slab at the mobility edge. The correct scaling law  $T \propto 1/L^2$  is obtained by solving SC equations with a position dependent  $D(\mathbf{r})$ .

The computations presented in this paper were performed on the cluster HealthPhy (CIMENT, Grenoble). S.E.S. acknowledges financial support from the French ANR (project No. 06-BLAN-0096 CAROL) and the French Ministry of Education and Research.

- 
- [1] S.K. Cheung and Z.Q. Zhang, *Phys. Rev. B* **72**, 235102 (2005).
  - [2] D. Vollhardt and P. Wölfle, *Phys. Rev. B* **22**, 4666 (1980); in *Electronic Phase Transitions* (Elsevier Science, Amsterdam, 1992), p. 1.
  - [3] E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, *Phys. Rev. Lett.* **42**, 673 (1979).
  - [4] B.A. van Tiggelen, A. Lagendijk, and D.S. Wiersma, *Phys. Rev. Lett.* **84**, 4333 (2000).
  - [5] S.E. Skipetrov and B.A. van Tiggelen, *Phys. Rev. Lett.* **92**, 113901 (2004).
  - [6] S.E. Skipetrov and B.A. van Tiggelen, *Phys. Rev. Lett.* **96**, 043902 (2006).
  - [7] N. Cherroret and S.E. Skipetrov, *Phys. Rev. E* **77**, 046608 (2008).
  - [8] C. Tian, *Phys. Rev. B* **77**, 064205 (2008).
  - [9] S.K. Cheung, X. Zhang, Z.Q. Zhang, A.A. Chabanov, and A.Z. Genack, *Phys. Rev. Lett.* **92**, 173902 (2004).
  - [10] In contrast to Ref. [1], we use a 2D cutoff  $q_{\text{max}} = \mu/\ell$  in the integration over momentum  $\mathbf{q}$  in Eq. (2) of Ref. [6] and adjust  $\mu = 1/3$  to obtain the mobility edge in the infinite medium at  $k\ell = 1$ . The need for a cutoff arises from the failure of the small- $q$  approximation implied by Eq. (1) of Ref. [6] for  $q > 1/\ell$  (small distances). The exact way of applying the cutoff (2D or 3D cutoff, exact value of  $\mu$ , etc.) does not affect scaling with (large)  $L$  as far as the cutoff is consistent with the definition of the mobility edge.